Bishop Domination on a Hexagonal Chess Board

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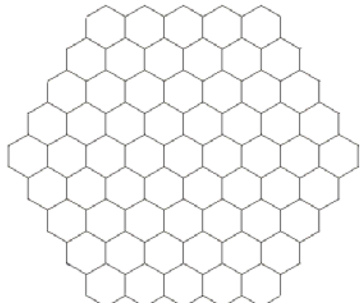
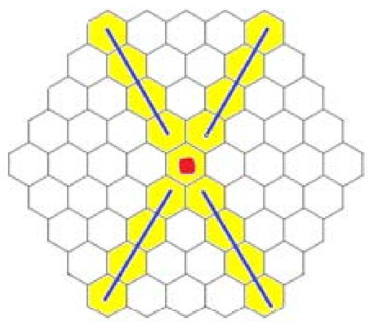
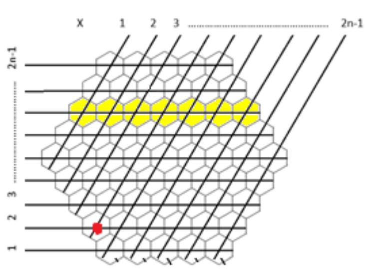
Hexagonal Chess is a modification of the traditional game of chess, played on a hexagonal board. A board is defined as an *n-board* if the board has *n* hexagons on the bottom row of the board. All boards we used for this game are regular hexagonal boards. This means that the board has the same number of hexagons on every side. Below is an example of a 5-board:



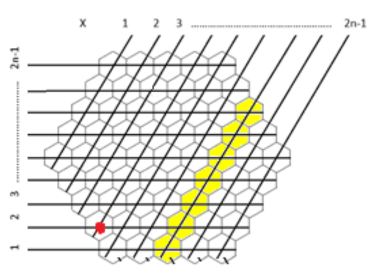
Traditional chess pieces are still used for this game of chess, however because of the shape of the spaces, the possible moves of each piece must be defined. In this proof, we considered only the bishop. The bishop, in this version of the game, may move in both diagonals, but not horizontally. For example, in the following graph, the red dot marks the placement of the bishop and the blue lines mark the spaces the bishop threatens:



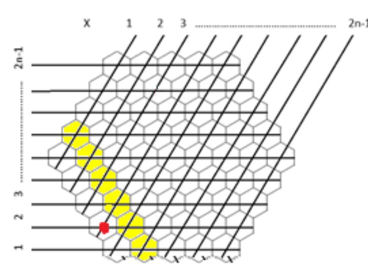
We are trying to develop a method to dominate an entire board with bishops given a board of any size. In order to do this, we defined a coordinate system. In this system, the x-axis is labeled on the top of the hexagon while the y axis is labeled down the side. Like the Cartesian coordinate system, y values change as you move up the graph. An example of this coordinate system is shown below:



y = 7



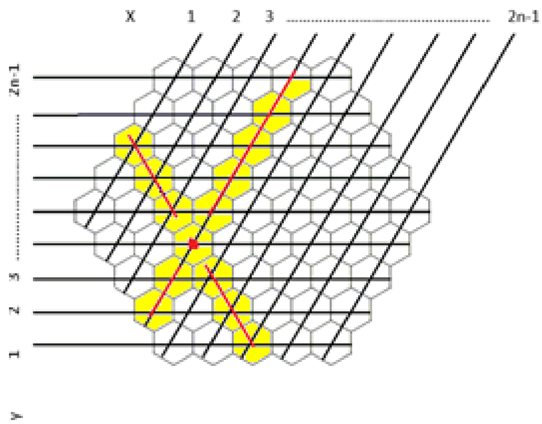
x = 7

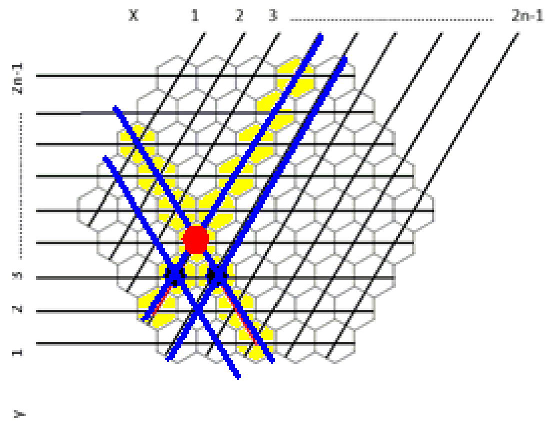


y = - x + 7

Coordinates can be represented with the ordered pair (x,y). For example, the red spot on the below images can be represented with the ordered pair (4,2). Also, on each of the below graphs, one of the three lines, x=7, y=7, or y=-x+7 is plotted, showing one of the three possible linear paths on the graph.

Lemma 1: If a bishop on space (a,b) threatens a region R, this same region R, as well as a new region is threatened by 2 bishops at either (a, b+1), (a+1, b+1) or at (a, b-1), (a-1, b-1). This is seen in the boards below:





The graph above on the left shows the region R threatened by one bishop on a spot (a,b).

In the board above on the right, the bishop on (a,b) has been replaced with two bishops on (a,b+1) and (a-1,b+1). The red region R is the common region threatened by both graphs. The blue region is the new region covered by two bishops that was not covered by one bishop. Likewise, replacing the bishop on (a,b) with two bishops on the spots below, (a,b-1) and (a+1,b-1).

Proof: Each bishop threatens space on two lines through the grid. If Bishop 1 is placed on space (a,b) , it threatens every space on the lines x=a and y= a+b-x.

If we replace Bishop 1 with Bishop 2 on space (a, b-1) and Bishop 3 on space (a+1, b-1), together then threaten the spaces on the 4 lines

Bishop 2 : $\begin{cases} x = a \\ y = a + b - 1 - x \end{cases}$ Bishop 3 : $\begin{cases} x = a + 1 \\ y = a + b - x \end{cases}$

If we instead replace Bishop 1 with Bishop 4 on space (a, b+1) and Bishop 5 on space (a-1, b+1), together then threaten the spaces on the 4 lines

Bishop 4 : $\begin{cases} x = a \\ y = a + b + 1 - x \end{cases}$ Bishop 5 : $\begin{cases} x = a - 1 \\ y = a + b - x \end{cases}$

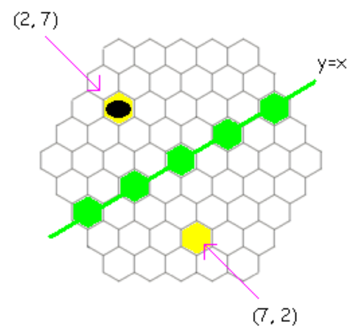
The sets of two new bishops always threatens the spacesa that Bishop 1 did.

QED, the region that one bishop covers is covered by replacing it with two bishops either directly above or directly below the original one.

Lemma 2: Any bishop on spot (a,b) threatens the spot (b,a).

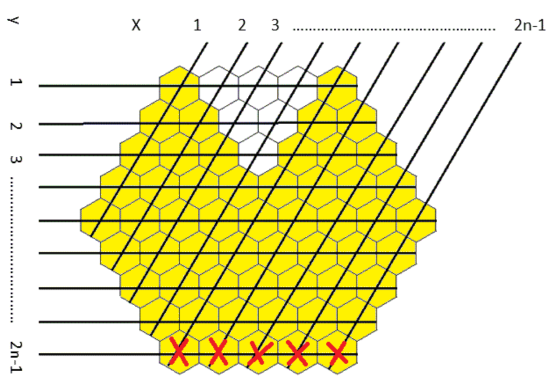
Proof: Bishop on (a,b) threatens all spots in the line y=a+b-x. Since (b,a) also satisfies this equation, we see that any bishop which threatens a point (a,b) also threatens its symmetric point (b, a). Note that the points (a, b) and (b, a) are reflections across y = x.

If we set (x,y)=(a,b), we get a=a+b-b. Therefore, a bishop on (a,b) always threatens the spot (b,a)
Looking above, we can also see that for any point (a,b) we choose, its symmetric point exists on the board.



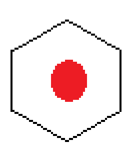
Theorem 1: Let T be the upper triangle, illustrated by the white region below, between and including the lines x=n-1 and y= -x+1. Then, the remainder of the graph (excluding T), can be covered by filling the base of the hexagon with bishops (placing a bishop on every spot in the line y= 2n-1).

The X's on the graph below show a sample set up as described above that covers everything except the triangle at the top of the graph.



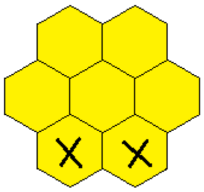
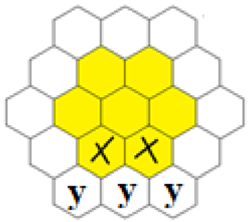
Proof: We shall proceed with a proof by induction. Let P_n be the statement that in an n-hexagon, a row of bishops on the base (y=2n-1) will cover every spot except the upper triangle T.

Base Case: n=1

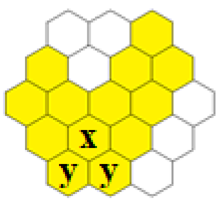


In the n-1 graph all the hexagons excluding the triangle are covered by bishops on the y=1 row from $n' \leq x' \leq 2n' - 1$

In the n graph, these are on the y=2 row from $n \leq x \leq 2n - 2$, as shown below. The x's mark spots that contain bishops. Note that on the n graph, the old bishops are now on the y=2 row.

Each of the spots covered by these bishops can be covered by two bishops diagonally down from the spot. These new bishops are shown with y's above. Also, the regions covered by any two adjacent bishops will overlap, and each will cover the same spot on the row y=2. For example, in the picture below the bishops marked by y's both cover the same spot on the row y=2, marked by an x.



So we will have to add one more bishop on the bottom row of the n graph, in order to cover the entire bottom row of the n graph.

These new bishops on the bottom row of the n graph will be on y=1 ranging from $n \leq x \leq 2n - 1$ and will cover all the hexagons in the n graph that were covered in the n-1 graph, as shown in the picture above.

Now that we have everything from the old graph (n-1) and the triangle covered, all we have are the new spots as shown below:

On line:	From, to:	Referenced by:
x=1	$1 \leq y \leq n$	Fig. 1
y=2n-1	$n \leq x \leq 2n-1$	Fig. 2
x=2n-1	$n \leq y \leq 2n-1$	Fig. 3
y=x+n-1	$1 \leq x \leq n$	Fig. 4
y=x-n+1	$n \leq x \leq 2n-1$	Fig. 5

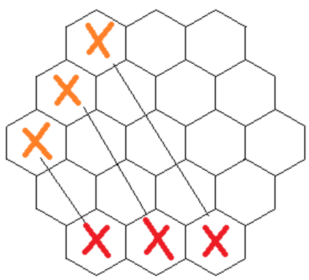


Figure 1

We have bishops on (1,x) where $n \leq x \leq 2n-1$.

Each one threatens the space (x,1), where $n \leq x \leq 2n-1$.

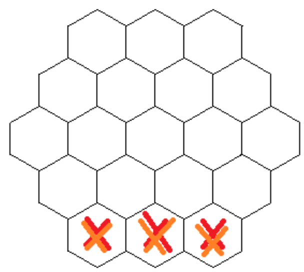


Figure 2

The bishop at spot (1,n) threatens all spots on the line x=1.

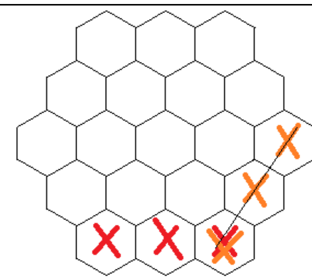


Figure 3

The bishop on (1, 2n-1) covers all spots on the line x=2n-1.

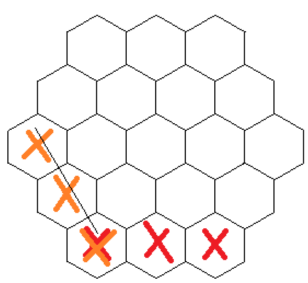


Figure 4

It also covers all spots on the line y=1+n-x.

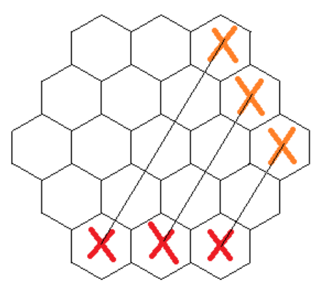


Figure 5

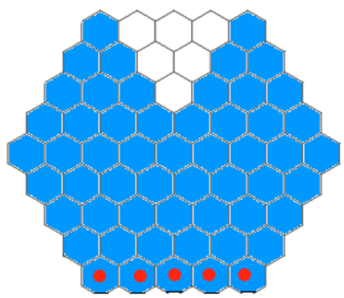
Since there is a bishop on every x line from $n \leq x \leq 2n-1$ which covers any space with an x value $n \leq x \leq 2n-1$ and goes on the line y=3n-1-x from $n \leq x \leq 2n-1$.

Thus, if P_n is true for any graph G, P_{n+1} must also be true. By induction, covering the bottom row of any graph G will cover all spots with the exception of the triangle on the top of the hexagon.

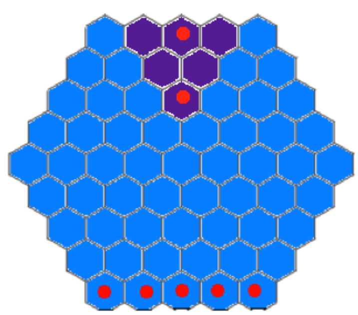
The Triangle Theorem:

From our previous set-up of bishops (placing bishops entirely across the bottom row of the board), with a sufficiently large hexagonal board, a triangle of spaces **not** covered by the row of bishops along the bottom will always be present.

Here is an example of the triangle formed on the 5-hexagon game. Notice the base of the triangle formed is of length 3, and in general the base of the triangle formed on an n-hexagon game will be of length n-2, since the bottom row of bishops will dominate the two spaces (1,1) and (2n-1,1) of the top row. The blue spaces represent spaces currently dominated by the bishops (red dots).

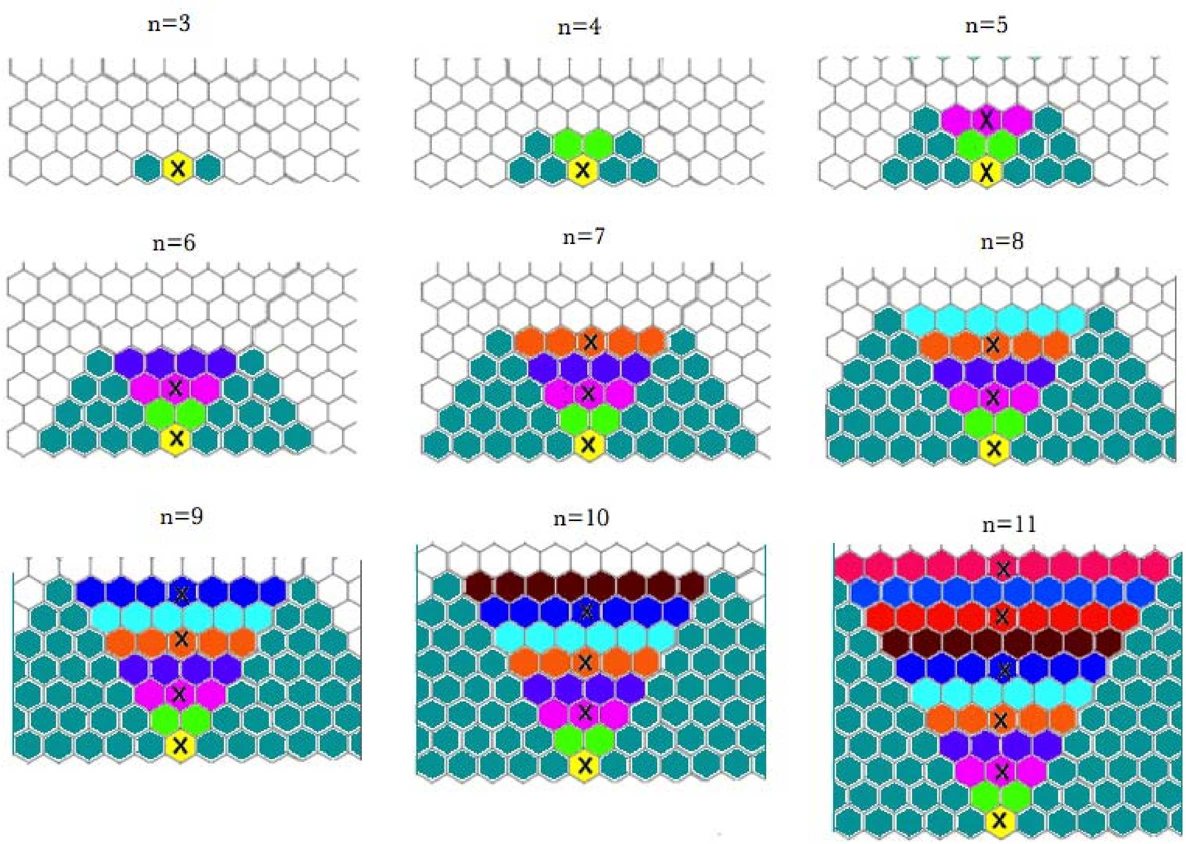


To cover the remaining triangle, two bishops are necessary, where the purple spaces are the spaces of the triangle dominated by the two new bishops.



The triangle theorem can be stated as follows:

For a game of hexagonal chess on an n-board in which the entire bottom row of the hexagon is covered with bishops, the un-dominated triangle can always be dominated by $\left\lceil \frac{n}{2} - 1 \right\rceil$ bishops by placing one bishop at the vertex of the triangle (near the center of the board), and then one bishop in the center space of every other row as follows (the teal region outside of the triangle represents spaces on the hexagon already dominated by the bottom row of bishops):



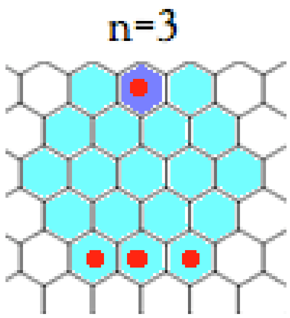
From this setup, it should be clear that bishops would always be placed on even rows for games on even n-boards, and on odd rows for odd n-boards. This is because the first bishop is always placed on the row two above the center row (since the coverage of the two corner bishops in the bottom row intersect at the point (k, k) so we will need the first bishop two y-values greater), which is the line $y = k + 2$ the second is on the line $y = k + 4$ the third on $y = k + 6$, etc. If k is even, these values are always even. If k is odd, these values are always odd.

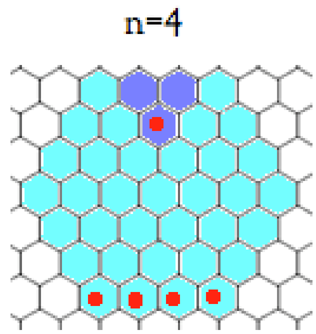
We will proceed by induction on the size of the hexagonal board.

Let P_n be the statement that for a size n hexagonal board, the number of bishops required to cover the uncovered triangle is $\left\lceil \frac{n}{2} - 1 \right\rceil$ and can be done so by placing one bishop at the vertex of the triangle closest to the center of the hexagon, and then placing one bishop in the center space in every other row as described above.

The simplest hexagonal chess game in which a triangle exists in with $n = 3$ with the "triangle" really just being one space. Therefore the entire triangle can be dominated with $\left\lceil \frac{3}{2} - 1 \right\rceil = 1$ bishop on this space.

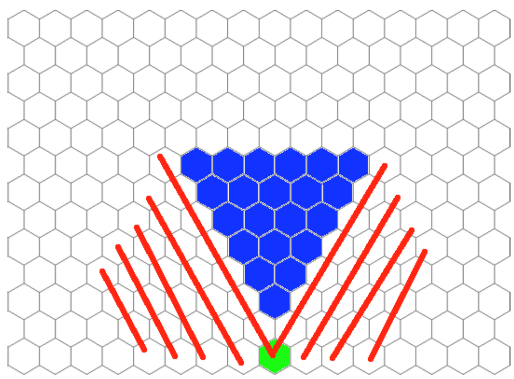
Therefore, P_3 is true. P_4 is also true, since a triangle of three spaces is dominated by $\left\lceil \frac{4}{2} - 1 \right\rceil = 1$ bishop, which can be placed on the top vertex of the triangle.





Consider the game on a board of size k.

From our described setup, bishops have been placed along the bottom row entirely. The first place these bishops will not dominate will be after they intersect paths. Here is an example for $k = 8$. Here the green space is where the bishops intersect paths, and the blue is the un-dominated triangle.



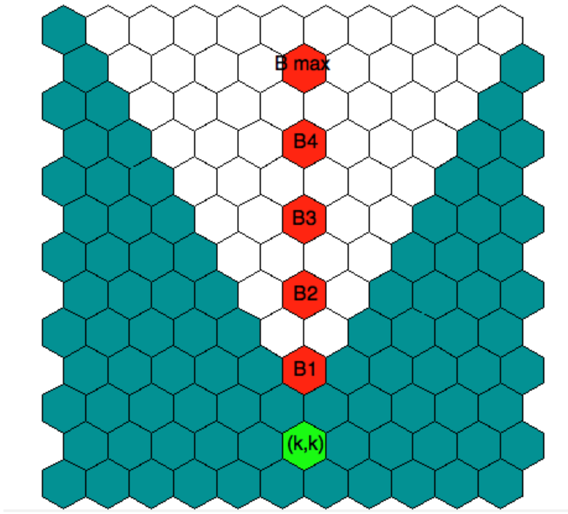
Since the equation of these two lines that intersect on this green point are known, the coordinates for all the bishops in the triangle are known.

The equation of the line coming from the bottom left corner is simply the line $x = k$ and the equation of the line coming from the bottom right is the line with slope -1 that passes through the bottom left point $(2k - 1, 1)$ which is simply $y = 2k - x$. These two lines intersect exactly at the point (k, k). From the prescribed setup, the first bishop is at one x-value less than this point and two y-values greater, the second is at one x-value less than the first and two y-values greater than the first, etc.

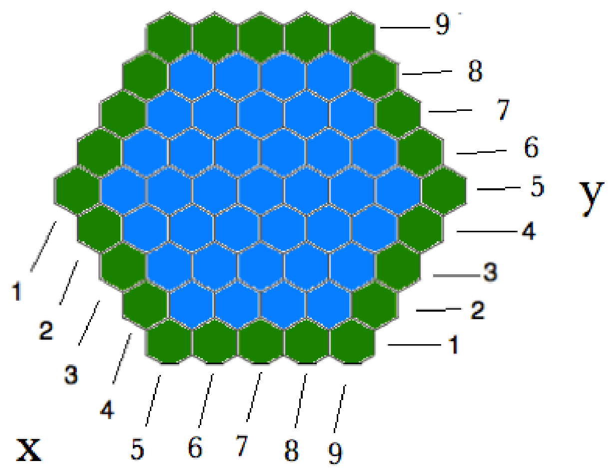
$B_1 = (k - 1, k + 2)$, $B_2 = (k - 2, k + 4)$, $B_3 = (k - 3, k + 6)$

And in general,

$B_i = (k - i, k + 2i)$



Now consider what happens to a general coordinate when a “ring” of hexagons is added to the board.



As one can see, the line that was previously $x = 1$ in the blue board is now $x = 2$, what was $x = 2$ is now $x = 3$, and in general, what was $x = a$ is now $x = a + 1$.

In the same way, all of the y coordinates of the old board are shifted up one unit when a ring is added. So in general, when a ring of hexagons is added to a board, the following shift takes place:

$(a, b) \rightarrow (a + 1, b + 1)$

Inductive step:

Assume P_k is true for some $k \in \mathbb{Z}^+$.

We will handle two specific cases, when k is odd and when k is even.

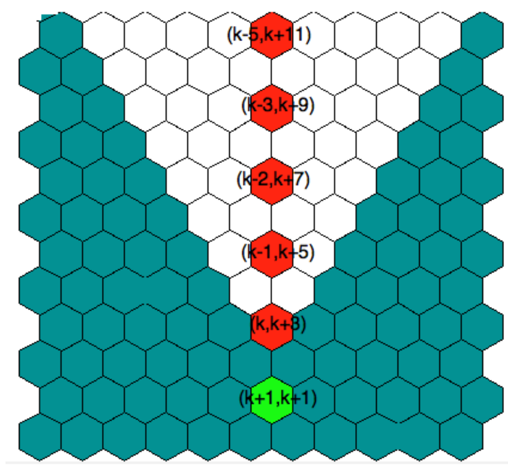
K IS EVEN:

Let's examine the case when k is even.

By assumption, P_k is true.

Consider the game on a board of size $k+1$, which is odd.

Since the intersection point discussed earlier from the two bishops in the bottom corners is now at the point $(k+1, k+1)$ the coordinates of all bishops desired to be placed is known. Using the one x-value less and two y-value greater reasoning:



$$B_1 = (k, k+3), B_2 = (k-1, k+5), B_3 = (k-2, k+7)$$

And in general,

$$B_i = (k - i + 1, k + 2i + 1)$$

Since the board is of odd size ($k+1$ is the size, which is odd), the top row of the board must contain a bishop, since the top row of the board is always an odd y-value (top row is $y = 2r - 1$ for board size r).

The coordinates of this "maximum" bishop, B_{max} are given by,

$$B_{max} = \left(\frac{k}{2} + 1, 2k + 1\right)$$

This coordinate is found by considering the following: the bishop is on the top row so its y coordinate is $y = 2(k+1) - 1 = 2k + 1$. It follows from the relation obtained from B_i that $k + 2i + 1 = 2k + 1$ and thus, $i = \frac{k}{2}$. Substituting this into the expression for the x-coordinate gives the above x-coordinate.

However, since this graph was created by adding a "ring" of hexagons to the even board of size k, the coordinates of the bishops from the even k-board can be found if they were to remain in place on the $k+1$ board by applying a shift to each of their coordinates.

In the even k-board, the intersection point from the two bottom corner bishops is (k, k) and again, using the one x-value less and two y-value greater reasoning, all of the coordinates of these bishops are known.

$N_1 = (k-1, k+2)$, $N_2 = (k-2, k+4)$, $N_3 = (k-3, k+6)$

And in general,

$$N_i = (k - i, k + 2i)$$

Since this was an even sized board, there was **not** a bishop on the top row. Therefore, the maximum bishop was on the next to last row, which means it had a y-coordinate of $2k - 2$ (since the top row is the line $y = 2k - 1$). This gives the coordinates of the maximum bishop of the k-board to be:

$$N_{max} = \left(\frac{k}{2} + 1, 2k - 2\right)$$

Applying the shift that occurs whenever you add a ring, each x and y coordinate should increase by a value of one. Therefore in the $k+1$ graph, the coordinates of the bishops from the k graph will be:

$$B_1^{\prime} = (k, k+3)$$

$$B_2^{\prime} = (k-1, k+5)$$

And in general,

$$B_i^{\prime} = (k - i + 1, k + 2i + 1)$$

Also,

$$B_{max}^{\prime} = \left(\frac{k}{2} + 2, 2k - 1\right)$$

Notice that all of the coordinates of these bishops are the required coordinates of the bishops on the $k+1$ graph, **except** that we need one more bishop to cover the new top space.

The coordinates of this bishop should be one x-value less and two y-values greater than what was the maximum, giving the following coordinates, which are exactly what was desired.

$$B_{actual\ max} = \left(\frac{k}{2} + 1, 2k + 1\right)$$

Notice that all of the coordinates of the bishops desired for the $k + 1$ board are the same as those shifted from the k board, except one more is needed. Therefore, if m bishops were needed for the k board, $m + 1$ will be needed for the $k + 1$ board (keep in mind we are only handling the case when k is even).

Since $k = 2a$ for some positive integer a,

$$\#Bishops\ required\ for\ k - board = \left\lceil \frac{2a}{2} - 1 \right\rceil = a - 1 = m$$

$$\#Bishops\ for\ k + 1\ board = \left\lceil \frac{2a + 1}{2} - 1 \right\rceil = a = m + 1$$

Thus a successful setup of bishops on the even k board implies a successful setup of bishops on the odd $k + 1$ board and satisfies the theorem.

K IS ODD:

Now we must consider if k were odd.

Again all of the bishops can be determined from the reference intersection point (k, k).

$$B_1 = (k - 1, k + 2), B_2 = (k - 2, k + 4), B_3 = (k - 3, k + 6)$$

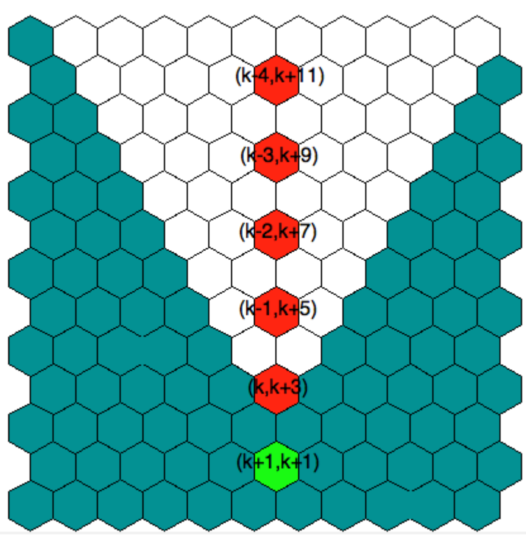
And in general,

$$B_t = (k - t, k + 2t)$$

Since k is odd, we need a bishop on the top row, which would have a y-coordinate of $2k - 1$. This gives the coordinates of the maximum bishop:

$$B_{max} = \left(\frac{k + 1}{2}, 2k - 1\right)$$

Now consider the even-sized, $k + 1$-board. Again the desired bishops are known based on the reference intersection point $(k + 1, k + 1)$.



$$N_1 = (k, k+3)$$

$$N_2 = (k-1, k+5)$$

And in general,

$$N_i = (k-i+1, k+2i+1)$$

Since the $k+1$ board is of even size, the top row will not have a bishop, so the next to last row will have a bishop that has the following coordinates:

$$N_{max} = \left(\frac{k+3}{2}, 2k\right)$$

Now consider the shift that occurs to all of the bishops when the ring of hexagons is added to make the $k+1$ board. All of the x-coordinates and y-coordinates from the k-board will gain one. The bishops then have the coordinates:

$$B_1' = (k, k+3)$$

$$B_2' = (k-1, k+5)$$

And in general,

$$B_i' = (k-i+1, k+2i+1)$$

Also,

$$B_{max}' = \left(\frac{k+3}{2}, 2k\right)$$

Notice that all of the bishops in the set B' satisfy all of the bishops required (those from set N).

Since k is odd, $k = 2a+1$ for some positive integer a.

The number of bishops for the k board and the $k+1$ board have been shown to be equal.

$$\#Bishops\ for\ k\ board = \left\lceil\frac{2a+1}{2} - 1\right\rceil = a$$

$$\#Bishops\ for\ k+1\ board = \left\lceil\frac{2a+2}{2} - 1\right\rceil = a$$

Thus a successful setup of bishops on an odd k board implies a successful setup on an even $k+1$ board and satisfies the theorem.

Since P_3, and $P_k \Rightarrow P_{k+1}$ for k odd, and since P_4 is true and $P_k \Rightarrow P_{k+1}$ for k even, by the Principle of Mathematical Induction, P_n is true for all integers greater than or equal to 3.