Path and Cycle Graphs in Passing Stones

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1 Introduction

In the game of *Passing Stones*, one begins with a graph and a certain number of stones placed on each vertex. During an iteration, each vertex simultaneously passes a stone to each of its adjacent vertices (neighbors) if and only if the vertex has at least as many stones as its degree.

We will observe the long-term behavior of path and cycle graphs with all stones initially on a single vertex, as well as establish two basic results on the periods of graphs that apply in general.

2 Definitions

We will define some terms and functions used in this paper.

Period - the number of iterations in the smallest block of repetitive behavior

Equilibrium - the state at which the next iteration will have the same configuration of stones as the current one; the onset of period 1 behavior

Saturation - a number of stones n such that $n \geq \sum \deg V$; this definition is motivated by the fact that if a saturation of stones is placed on a single vertex of a path graph, then the equilibrium will have at least one stone on every vertex

$P(n) =$ the number of iterations for a path graph of length n to reach equilibrium where a saturation of stones is placed at the one of the vertices of degree 1

$C(n) =$ the number of iterations for a cycle graph of length n to reach equilibrium where a saturation of stones is placed on a single vertex

3 General graphs

Theorem 1. Every graph has a finite period.

Proof. Suppose the period is infinite. Then there must be an infinitude of different arrangements to form the period. However, this is a contradiction because the number of possible arrangements of stones is finite. Thus every graph has a finite period. $\square$

Theorem 2. Any length period is attainable.

Proof. A period of 1 can be obtained by a graph consisting of a single vertex with 0 stones on it. A period of 2 can be obtained by a graph with two vertices connected to each other, one with 1 stone and one with 0 stones.

Periods greater than 2 can be obtained by a cycle graph with $1, 1, \ldots, 1, 2, 0$ stones on each vertex. Suppose such a graph has n vertices, for $n \geq 3$. Call the vertex with 2 stones V and the vertex with 0 stones W. During an iteration, vertex V gives a stone to each of its neighbors, and it gains no stones, so it will have 0 stones at the end of the iteration. Thus vertex W will gain a stone. Also, the vertex that is adjacent to V but not W will gain a stone from V, so it will have 2 stones at the end of the iteration. All other vertices are unaffected since they each have only 1 stone. Therefore, the new arrangement of stones is a cyclic permutation of the original graph, shifted one to the left. Since there are n such permutations, the period is n.

Thus, any possible period is obtainable by some graph. $\square$

4 Path graphs

Theorem 3. If we place a saturation of stones on a vertex k away from the farthest end vertex of an n-path, then equilibrium will be reached, and the number of iterations before equilibrium is independent of both n and the number of stones and equal to $P(k)$.

Proof. Suppose we have a path of length n, and we a saturation of stones on vertex A. Then for some integer $k < n$ we can say that there are k vertices to the left and $n - k - 1$ vertices to the right of the vertex A. By symmetry, we can assume that $k \geq n - k - 1$. Now if we consider the subgraph of path length $2n - 2k - 1$ centered around vertex A, then the right side with $n - k - 1$ vertices will reach equilibrium at the number of iterations defined by $P(n - k - 1)$. Upon reaching this state, all vertices to the right of A are at equilibrium, and since A contained a saturation of stones, it will not interfere with the vertices to the left of A. Therefore we can say that the behavior of stones on both subgraphs are independent. Since $P(n)$ is a strictly increasing function, $P(k) \geq P(n - k - 1)$ and $P(k)$ is the number of iterations it takes for this graph to reach equilibrium. $\square$

5 An important result on equilibria of path graphs

We computed $P(n)$ for $1 \leq n \leq 700$ using a computer program, and we noticed that the growth of the function is approximately quadratic. In order to investigate this behavior, we computed the finite differences $P_1(n) = P(n+1) - P(n)$ for $1 \leq n \leq 699$ and the second finite differences $P_2(n) = P_1(n+1) - P_1(n)$ for $1 \leq n \leq 698$.

Interestingly, we found that $|P_2(n) - 3| \leq 1$ for all values of n tested. This means that the growth of $P(n)$ is bounded between those of a quadratic and a cubic function.

In an attempt to find a pattern in the values of $P_2(n)$, we computed the values of k for which $P_2(k) = 3$. The first few such k are

$$k = 1, 2, 3, 4, 8, 9, 20, 21, 49, 50, 119, 120, 288, 289, 696, 697.$$

We noticed that the values of k come in pairs of consecutive integers, as in

$$(1, 2), (3, 4), (8, 9), (20, 21), (49, 50), (119, 120), (288, 289), (696, 697).$$

Considering the first value of each pair, we obtain the following sequence, which we will call $\{Q_n\}$:

$$1, 3, 8, 20, 49, 119, 288, 696.$$

Taking finite differences of adjacent terms, we get

$$2, 5, 12, 29, 70, 169, 408.$$

These numbers are precisely the *Pell numbers*, a sequence of numbers $\{P_n\}$ defined recursively by $P_0 = 0$, $P_1 = 1$, and

$$P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2.$$

A closed-form expression for $\{P_n\}$ is

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

Therefore, the sequence $\{Q_n\}$ simply contains the partial sums of P_n,

$$Q_n = \sum_{i=1}^{n} P_i.$$

A closed-form expression for $\{Q_n\}$ is

$$Q_n = \frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1} - 2}{4}.$$

Let Q_n^{-1} denote the inverse function of Q_n.

We also noticed a pattern in those values of k for which $P_2(k) = 2$ and those for which $P_2(k) = 4$. Letting q be the largest value less than k that is in the sequence $\{Q_n\}$ (where n is an integer), we considered finite differences of the sequence $\left\{\left\lfloor\sqrt{2}(k-q)\right\rfloor\right\}_{k\in\mathbb{N}}$.

We found that $P_2(k) = 2$ whenever

$$\left\lfloor \sqrt{2}(k-q) \right\rfloor - \left\lfloor \sqrt{2}(k-q-1) \right\rfloor = 1,$$

and that $P_2(k) = 4$ whenever

$$\left\lfloor \sqrt{2}(k-q) \right\rfloor - \left\lfloor \sqrt{2}(k-q-1) \right\rfloor = 2.$$

Redefining k as n, since $\lfloor Q_n^{-1} \rfloor$ is the index of the largest value of $\{Q_n\}$ less than n, we have $q = Q_{\lfloor Q_n^{-1} \rfloor}$. Therefore, the above can be combined into a single expression for $P_2(n)$:

$$2\left(\left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor}\right) \right\rfloor - \left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right) \right\rfloor \right).$$

Combining these results leads to a conjecture on the second finite differences of $P(n)$:

Conjecture 4.

$$P_2(n) = \begin{cases} 3, & \text{if } n - Q_{\lfloor Q_n^{-1} \rfloor} \leq 1 \\ 2\left(\left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor}\right) \right\rfloor - \left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right) \right\rfloor \right), & \text{otherwise} \end{cases}$$

To find an expression for $P_1(n)$, we rewrite the function in terms of $P_2(n)$:

$$\begin{aligned} P_1(n) &= P_1(1) + \sum_{i=1}^{n-1} P_1(i+1) - P_1(i) \\ &= P_1(1) + \sum_{i=1}^{n-1} P_2(i) \\ &= 1 + \sum_{i=1}^{n-1} P_2(i). \end{aligned}$$

We take a constructive approach to finding the partial sum $\sum_{i=1}^{n-1} P_2(i)$. Split the n terms in the sum into groups $Q_k \leq i \leq Q_{k+1} - 1$, for integers $k \geq 1$. Call the kth group *complete* if all terms in the range $Q_k \leq i \leq Q_{k+1} - 1$ are present, and *incomplete* otherwise.

The final group, corresponding to $k = \lfloor Q_{n-1}^{-1} \rfloor$, contains the terms $Q_{\lfloor Q_{n-1}^{-1} \rfloor} \leq i \leq n-1$, and if $Q_{\lfloor Q_{n-1}^{-1} \rfloor} = Q_{\lfloor Q_n^{-1} \rfloor}$, then this group will be incomplete. Thus there are $\lfloor Q_{n-1}^{-1} \rfloor$ total groups and $\lfloor Q_n^{-1} \rfloor - 1$ complete groups.

Taking care of complete groups first, we look at an arbitrary group

$$\sum_{i=Q_k}^{Q_{k+1}-1} P_2(i).$$

Two of the terms in this group, $i = Q_k$ and $i = Q_k + 1$, will be equal to 3, according to Conjecture 4; together they contribute 6 to the sum. The sum of the other terms are, based on Conjecture 4,

$$\sum_{i=Q_k+2}^{Q_{k+1}-1} P_2(i) = 2 \sum_{i=Q_k+2}^{Q_{k+1}-1} \left\lfloor \sqrt{2}\,(i - Q_k) \right\rfloor - \left\lfloor \sqrt{2}\,(i - Q_k - 1) \right\rfloor .$$

Notice that the sum telescopes to

$$\begin{aligned}\sum_{i=Q_k+2}^{Q_{k+1}-1} P_2(i) &= 2\left(\left\lfloor \sqrt{2}(Q_{k+1} - Q_k - 1) \right\rfloor - \left\lfloor \sqrt{2} \right\rfloor \right) \\ &= 2\left(\left\lfloor \sqrt{2}(P_{k+1} - 1) \right\rfloor - \left\lfloor \sqrt{2} \right\rfloor \right) \\ &= 2\left(\left\lfloor \sqrt{2}P_{k+1} - \sqrt{2} \right\rfloor - 1 \right).\end{aligned}$$

Now, the Pell numbers P_k have the special property that $\sqrt{2}P_{k+1} \approx P_{k+1} + P_k$ for all $k \geq 1$. The accuracy of this approximation is due to the small magnitude of $(1-\sqrt{2})^k$ in the closed-form expression for $\{P_n\}$. For $k \geq 1$, the maximum possible deviation of $\sqrt{2}P_{k+1}$ from an integer is

$$\left|(1-\sqrt{2})^2\right| = 3 - 2\sqrt{2}.$$

Since both $\sqrt{2} + (3 - 2\sqrt{2})$ and $\sqrt{2} - (3 - 2\sqrt{2})$ are between 1 and 2, we have

$$\left\lfloor \sqrt{2}P_{k+1} - \sqrt{2} \right\rfloor = P_{k+1} + P_k - 2.$$

Therefore,

$$\sum_{i=Q_k+2}^{Q_{k+1}-1} P_2(i) = 2(P_{k+1} + P_k - 3).$$

Thus the kth complete group contributes to $P_1(n)$ the value

$$\sum_{i=Q_k}^{Q_{k+1}-1} P_2(i) = 6 + 2(P_{k+1} + P_k - 3).$$

Summing across the $\lfloor Q_n^{-1} \rfloor - 1$ complete groups, corresponding to $1 \leq k \leq \lfloor Q_n^{-1} \rfloor - 1$, gives a total contribution of

$$\begin{aligned}\sum_{k=1}^{\lfloor Q_n^{-1} \rfloor - 1} 6 + 2(P_{k+1} + P_k - 3) &= 2 \sum_{k=1}^{\lfloor Q_n^{-1} \rfloor - 1} P_{k+1} + P_k \\ &= 2\left(-1 + \sum_{k=1}^{\lfloor Q_n^{-1} \rfloor} P_k + P_{k-1} \right)\end{aligned}$$

from complete groups. We will show by induction that

$$\sum_{k=1}^{n} P_k + P_{k-1} = P_{n+1} - 1 \text{ for all } n \geq 1.$$

For the base case $n = 1$, we have

$$\sum_{k=1}^{1} P_k + P_{k-1} = P_1 + P_0 = P_2 - 1 = 1,$$

as desired. Now, suppose that $\sum_{k=1}^{n} P_k + P_{k-1} = P_{n+1} - 1$ for some n. We have

$$\begin{aligned}\sum_{k=1}^{n+1} P_k + P_{k-1} &= P_{n+1} + P_n + \sum_{k=1}^{n} P_k + P_{k-1} \\ &= P_{n+1} + P_n + P_{n+1} - 1 \\ &= 2P_{n+1} + P_n - 1 \\ &= P_{n+2} - 1.\end{aligned}$$

This completes the proof. Substituting gives $2\left(P_{\lfloor Q_n^{-1} \rfloor + 1} - 2\right)$ as the total contribution from complete groups.

Now, we consider the incomplete group $k = \lfloor Q_{n-1}^{-1} \rfloor$, assuming that it exists. If such a group exists, then $k = \lfloor Q_{n-1}^{-1} \rfloor = \lfloor Q_n^{-1} \rfloor$.

If $n - Q_{\lfloor Q_n^{-1} \rfloor} \leq 2$, then the contribution from the incomplete group will solely consist of $\left(n - Q_{\lfloor Q_n^{-1} \rfloor}\right)$ 3's, for a total contribution of

$$3\left(n - Q_{\lfloor Q_n^{-1} \rfloor}\right).$$

If $n - Q_{\lfloor Q_n^{-1} \rfloor} > 2$, then there will be a contribution of 6 from the two 3's, and based on Conjecture 4, the other terms sum to

$$\begin{aligned}\sum_{i=Q_{\lfloor Q_n^{-1} \rfloor}+2}^{n-1} P_2(i) &= 2 \sum_{i=Q_{\lfloor Q_n^{-1} \rfloor}+2}^{n-1} \left\lfloor \sqrt{2}\left(i - Q_{\lfloor Q_n^{-1} \rfloor}\right) \right\rfloor - \left\lfloor \sqrt{2}\left(i - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right) \right\rfloor \\ &= 2\left(\left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right) \right\rfloor - \left\lfloor \sqrt{2} \right\rfloor\right) \\ &= 2\left(\left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right) \right\rfloor - 1\right).\end{aligned}$$

Therefore, for incomplete groups, the terms sum to

$$\begin{cases} 3(n - Q_{\lfloor Q_n^{-1} \rfloor}), & \text{if } n - Q_{\lfloor Q_n^{-1} \rfloor} \leq 2 \\ 6 + 2\left(\left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right) \right\rfloor - 1\right), & \text{otherwise} \end{cases}$$

Substituting contributions from complete and incomplete groups into $P_1(n) = 1 + \sum_{i=1}^{n-1} P_2(i)$ gives us a corollary for the first finite differences of $P(n)$:

Corollary 5.

$$P_1(n) = 2P_{\lfloor Q_n^{-1} \rfloor+1} - 3 + \begin{cases} 3(n - Q_{\lfloor Q_n^{-1} \rfloor}), & \text{if } n - Q_{\lfloor Q_n^{-1} \rfloor} \leq 2 \\ 4 + 2\left(\left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right)\right\rfloor\right), & \text{otherwise} \end{cases}$$

This can also be written as

$$P_1(n) = 2P_{\lfloor Q_n^{-1} \rfloor+1} - 3 + \min\left(3(n - Q_{\lfloor Q_n^{-1} \rfloor}),\ 4 + 2\left(\left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right)\right\rfloor\right)\right).$$

Now, we rewrite $P(n)$ in terms of $P_1(n)$:

$$P(n) = \sum_{i=1}^{n-1} P_1(i).$$

We first evaluate $\sum_{i=1}^{n-1} P_{\lfloor Q_n^{-1} \rfloor+1}$. Partitioning the terms into groups $Q_k \leq i \leq Q_{k+1} - 1$, we see that the kth complete group has $Q_{k+1} - Q_k = P_{k+1}$ terms, each with value P_{k+1}, so complete groups contribute a sum of

$$\sum_{i=1}^{\lfloor Q_n^{-1} \rfloor - 1} P_{k+1}^2 = -1 + \sum_{i=1}^{\lfloor Q_n^{-1} \rfloor} P_k^2.$$

We will prove inductively that

$$\sum_{i=1}^{n} P_k^2 = \frac{P_n P_{n+1}}{2} \text{ for all } n \geq 1.$$

The base case $n = 1$ gives

$$P_1^2 = \frac{P_1 P_2}{2} = 1.$$

Suppose that $\sum_{i=1}^{n} P_k^2 = \frac{P_n P_{n+1}}{2}$ for some n. Then

$$\begin{aligned} \sum_{i=1}^{n+1} P_k^2 &= P_{n+1}^2 + \sum_{i=1}^{n} P_k^2 \\ &= P_{n+1}^2 + \frac{P_n P_{n+1}}{2} \\ &= \frac{P_{n+1}\left(2P_{n+1} + P_n\right)}{2} \\ &= \frac{P_{n+1} P_{n+2}}{2}, \end{aligned}$$

completing the proof. Therefore complete groups contribute a sum of

$$\frac{P_{\lfloor Q_n^{-1} \rfloor} P_{\lfloor Q_n^{-1} \rfloor+1}}{2} - 1.$$

Similarly, if an incomplete group exists, then it has $n - Q_{\lfloor Q_n^{-1} \rfloor}$ terms, each with value $P_{\lfloor Q_n^{-1} \rfloor + 1}$. Thus

$$\sum_{i=1}^{n-1} P_{\lfloor Q_n^{-1} \rfloor + 1} = \frac{P_{\lfloor Q_n^{-1} \rfloor} P_{\lfloor Q_n^{-1} \rfloor + 1}}{2} + \left(n - Q_{\lfloor Q_n^{-1} \rfloor}\right) P_{\lfloor Q_n^{-1} \rfloor + 1} - 1.$$

This leads to the following corollary on $P(n)$:

Corollary 6. Let $M(n) = \min\left(3(n - Q_{\lfloor Q_n^{-1} \rfloor}),\ 4 + 2\left(\left\lfloor \sqrt{2}\left(n - Q_{\lfloor Q_n^{-1} \rfloor} - 1\right)\right\rfloor\right)\right)$. We have

$$P(n) = P_{\lfloor Q_n^{-1} \rfloor} P_{\lfloor Q_n^{-1} \rfloor + 1} + 2\left(n - Q_{\lfloor Q_n^{-1} \rfloor}\right) P_{\lfloor Q_n^{-1} \rfloor + 1} - 3n + 1 + \sum_{i=1}^{n-1} M(i).$$

Although this is not in closed form, it provides a relatively easy way to compute values of $P(n)$. The following sequence gives $P(n)$ for $1 \leq n \leq 20$:

$$0, 1, 5, 12, 22, 35, 50, 69, 90, 114, 141, 170, 203, 238, 277, 318, 361, 408, 457, 510$$

A statistical approach provides a different perspective on the growth of $P(n)$. A quadratic fit was applied to the 700 points $(n, P(n))$, where $1 \leq n \leq 700$. The resulting quadratic, with uncertainties included, has equation

$$\widehat{P(n)} = (1.4142128 \pm 0.0000006)n^2 - (2.8281 \pm 0.0004)n + (0.79 \pm 0.06)$$

Evidently, $P(n)$ is very close to $\sqrt{2}n^2 - 2\sqrt{2}n$. For $1 \leq n \leq 700$, $P(n)$ never deviates from $\sqrt{2}n^2 - 2\sqrt{2}n$ by more than about 2.3. The near-perfect fit of the quadratic to $P(n)$ strongly supports that the growth of $P(n)$ is $\mathcal{O}(n^2)$.

6 Cycle graphs

Conjecture 7. If we place k stones ($k \geq 2n$) on vertex V of an n-cycle, then the final behavior will be period 1 with $k - 2n + 2$ stones on vertex V and 2 stones on every other vertex.

Conjecture 8. If we place n stones on a vertex of an n-cycle, then the equilibrium will be:

- Odd n: Period 1 with 1 stone on each vertex
- Even n: Period 2 with $2, 0, 2, 0, \ldots$ and $0, 2, 0, 2, \ldots$ stones on each vertex

7 An important result on equilibria of cycle graphs

Theorem 9.

$$C(n) = \begin{cases} P(\frac{n}{2} + 1), & \text{if } n \equiv 0 \pmod 2 \\ P(\frac{n+3}{2}) - n, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Proof. We prove the two cases for n separately.

Case 1: $n \equiv 0 \pmod 2$. If we have a cycle of even length n with a saturation of stones placed on some vertex A, then the graph will exhibit behavior analogous to that of a path of length $\frac{n}{2}+1$ with a saturation of stones on the center vertex A', up until the two furthest vertices are reached by stones. When this occurs, we expect the behavior to diverge because the edge vertices pass stones right after they receive them. In the cycle graph, however, a vertex only passes stones once it has received two or more from its neighbors.

However, if we consider the vertex diametrically opposite to A to be the combination of the two edge vertices in the path graph, then due to symmetry, the behavior is consistent. This is because symmetry demands that this vertex receive either 0 or 2 stones, and upon receiving stones it will immediately pass them out in the same manner as the path graph. Therefore, $C(n)$ is equivalent to the number of iterations until equilibrium for a path of length $\frac{n}{2}+1$ with a saturation of stones on A', or $P(\frac{n}{2}+1)$.

Case 2: $n \equiv 1 \pmod 2$. If we have a cycle of odd length n, then we also have symmetry relating to a path graph of length $\frac{n+3}{2}$. This behavior occurs if we consider the edge vertices on the path graph to be combined with the second vertex of the opposite side. Let us define the edge vertices of the path graph to be B and the second vertices to be B'. When B' has only one stone, it does not pass on the next iteration, a behavior which is identical for the path graph and its cycle graph analog. When B' has two stones, it will pass to both adjacent vertices, and since B only ever has 0 or 1 stones, their behaviors are independent even if the vertices are combined. Therefore, the path graph of length $\frac{n+3}{2}$ is a valid analog of the cycle graph of length n.

However, instead of converging at the same time as $P(\frac{n+3}{2})$, we have an extra n moves which are not included on the $C(n)$. Looking at the sequences for the path and cycle graphs to equilibrium, we discover that $C(n)$ achieves equilibrium at the same time that the path graph reaches an arrangement of $2, 2, \ldots, 2, 0$ stones. This behavior occurs because of the way the vertices of the path graph analog are overlapped.

Thus we would like to show that for a path graph of length n and initial state $2, 2, \ldots, 2, 0$, it takes $2n-3$ moves to resolve to equilibrium, thus proving that the excess iterations for $P(\frac{n+3}{2})$ is indeed n. From the sequence $2, 2, \ldots, 2, 0$ we see that the only stone which is out of place is the one in the leftmost vertex, C, and it needs to be moved to the right most vertex C'. In order to do this, $n-1$ iterations must pass for the effect of no stones on C' to reach C. In the process of moving this stone from C to C', we have disrupted all of the stones not on C and C', which are now in some pattern of 1 or 3 stones. Since it took one iteration to disrupt each vertex, it follows that it should take one iteration to resolve one of these vertices, leaving $n-2$ moves before we reach equilibrium. Thus, the total number of iterations is $2n-3$, and we have proved that for $P(\frac{n+3}{2})$, the excess number of moves is exactly n. Altogether this gives us $C(n) = P(\frac{n+3}{2}) - n$.

Therefore, combining the results of the two cases, we have a formula for $C(n)$ that relates to the function $P(n)$ for path graphs for all n. □

8 Incorrect results

This section contains results that we later proved to be false or not useful.

Conjecture 10 (false). The period of a graph is less than the maximum vertex degree.

Disproof. Consider two vertices connected to each other. Each has degree 1, but if we place one stone on a vertex, the graph oscillates between two positions with period 2, so the original statement is false. □

Result 11.

$$Q_n^{-1} \approx \frac{\log(4Q_n + 2)}{\log(1 + \sqrt{2})} - 1$$

Remark. This result comes from applying the approximation $(1 - \sqrt{2})^{n+1} \approx 0$ to the closed-form expression for Q_n. This gives

$$Q_n \approx \frac{(1 + \sqrt{2})^{n+1} - 2}{4},$$

and solving for n, we get the approximation for Q_n^{-1}. However, this estimate is not useful because it gives an incorrect value for $\lfloor Q_n^{-1} \rfloor$ for $n = Q_{2k}$, where k is an integer.

9 Further research

As Passing Stones is an unsolved question, there still remain many areas open to research. We have only considered path and cycle graphs with all stones initially on one vertex. However, the problem can be extended to general graphs with any number of stones on each vertex.

It would be interesting to search for an algorithm that can give the periodicity of a given graph and arrangement of stones. The ability to predict periodic behavior for a graph is also useful in the study of networks in the real world, as is the ability to predict how many iterations it takes to reach such periodic behavior. To tackle these questions, heuristics may have to be applied.

Another extension is to take a more statistical approach and suppose that stones are passed from a vertex to adjacent vertices with a certain probability given that the number of stones on the vertex exceeds its degree. This lack of determinism may be more representative of real-life situations.