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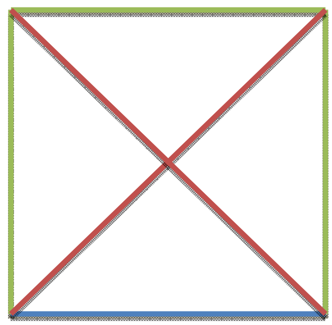
Positive Triangle Game

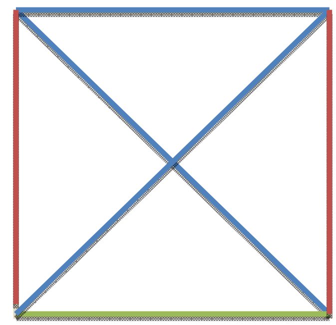
Two players take turns marking the edges of a complete graph K_n, for some n with (+) or (-) signs. The two players can choose either mark (this is known as a choice game). In this game, the goal or winning triangle can contain marks made by both players.

a) Under what conditions does the first player win?
b) If the first player to create a positive triangle loses, under what conditions does the first player win?
c) For which complete graphs is a draw (no winner) possible?

Definitions

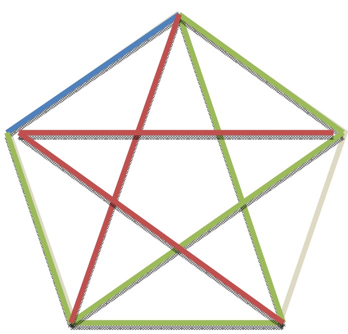
- Optimal strategy: labeling edges such that the opposing layer cannot win immediately in the following turn
- n: number of vertices
- Positive triangle (P.T.): a triangle where labeled edges have an even number of (-) labels.
- P_A: Player A
- P_B: Player B
- X_y: Player Xs Yth move.
- V_x: Vertex X, as labeled by supplement
- P_4: A Path graph on 4 vertices
- Supplement Notes
  - Edges are only drawn on supplemental graphs if those edges have been labeled in the actual game. This is to provide a clear and uncluttered timeline of what is occurring.
  - All vertices and edges are drawn WLOG
  - Edges labeled X_y mean Player Xs Yth move. Exact labels of edges are arbitrary, as explained in Part B Note 1
- Draw: A complete labeling with no positive triangles; no winner is established
- P-T-Wins: Positive-Triangle-Wins Game
- P-T-Loses: Positive Triangle-Loses Game
- Complex: a setup in which there is a cycle of four lines, three of which have the same sign, and one of which has a different sign. This cancels out 2 options for the players, as both a positive sign or a negative sign would create a positive triangle.
  - There are two forms where complexes can be seen:





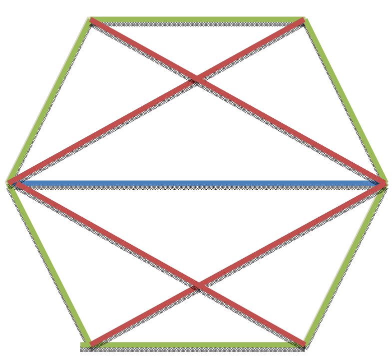
Green- Positive
Blue- Negative
Red- No longer safe moves

- Complexes are what prevent ties and sets up a winner for part b.
- In instances larger than $K_{4,}$ more than one complexes may occur. However, that does not mean four options will necessarily be cancelled out.



This double complex uses two of the same lines and both cancel out the same corner. This leaves only three options closed to the players.

There are 10 moves in $K_{5.}$ It took six moves to make the double complex. 3 more moves are taken away by the double complex. This means Player A will win.



Here, there are two complexes that only share one line. Since these two complexes do not overlap, four options are cancelled out rather than three.

In other cases, complexes are formed, but both of the options that would've been cancelled were already cancelled by other complexes. These are considered *arbitrary complexes.*

In this same way, in larger graphs there can be multiple complexes formed, and there is also a limit on how many can and will be formed in a games.

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Part A

Winning strategy for Player A and Prediction:

Theorem: When $\left\lfloor \frac{n}{2} \right\rfloor$ is even, P_B wins and if it is odd, P_A wins.

Note there are 2 ways to achieve P.T.: (-,-,+) or (+,+,+). Since (-,+), (-,-), and (+,+) would lead to a P.T. respectively with a (-), (+) or (+), with any combination of labels on 2 sides of a triangle, it will be possible to create a positive triangle with some third label.

Optimal strategy therefore dictates that players should not label an edge adjacent to an already labeled edge, because a positive triangle can and will be formed on the next turn by the opposing player.

Thus, specific labels become irrelevant and the game becomes a matter of preventing coincidence.

The maximum number of edges that are not coincident in any complete graph is $\left\lfloor \frac{n}{2} \right\rfloor$ at each turn. Each edge takes up 2 vertices and in a graph with n vertices, $\left\lfloor \frac{n}{2} \right\rfloor$ non-coincident edges can be drawn at a time.

Therefore, optimal strategy for each player will consist of labeling one of these non-coincident edges per turn. Once these non-coincident edges run out, the player will be forced to label an edge coincident to a labeled edge. Once this is done, the other player will be able to create a positive triangle and win. Therefore, whoever labels the last non-coincident edge wins. Thus, if $\left\lfloor \frac{n}{2} \right\rfloor$ is even, P_B labels the last edge and wins. If $\left\lfloor \frac{n}{2} \right\rfloor$ is odd, P_A wins.

Alternatively, when $\left\lceil \frac{\frac{n(n-1)}{2}}{2(n-2)+1} \right\rceil$ is even, P_A wins and if odd, P_B wins.

The total number of edges in a complete graph is $\frac{n(n-1)}{2}$ edges. Once the first edge is labeled by P_A, there are $2(n-2)$ edges that are coincident to that edge. The initial edge can also not be labeled again. Therefore, there are $2(n-2)+1$ edges that cannot be labeled by P_B during optimal play. Dividing the total edges by the number of edges unavailable to the next player gives the number of total moves in the game. If an even number results, P_A wins because the last winning move to create a positive triangle will be taken by him. If an odd number results, P_B wins

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Playing with Multiple Graphs:

Theorem: $\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor$; if odd, player A wins, if even player B wins.

Proof:

In this case, the optimal strategy is still to avoid labeling an edge coincident to an already labeled edge. This allows the next player to easily create a positive triangle.

As mentioned in *Part A Winning strategy and Prediction*, the number of non-coincident edges that can be drawn at a time in Graph 1 is $\left\lfloor \frac{n_1}{2} \right\rfloor$.

Likewise, the number of non-coincident edges that can be drawn at a time in Graph 2 is $\left\lfloor \frac{n_2}{2} \right\rfloor$

Immediately before a positive triangle is formed, both graphs will have the maximum number of non-coincident moves played.

Since this will be the length of the game, this therefore proves that the total number of edges, and therefore moves played, before sharing a common vertex is the sum of the moves for each individual graphs, $\left\lfloor \frac{n_1}{2} \right\rfloor + \left\lfloor \frac{n_2}{2} \right\rfloor$.

Thus, if this total is even, player A will play the last non-coincident edge, forcing player one to label a coincident edge, allowing player B to win. If the total is odd, player A will play the last non-coincident edge, guaranteeing the win.

This can be extended to m graphs, because the number of moves before one must play on a vertex provided is just the sum of all of the moves on every graph, or:

$$\sum_{i=1}^{m} \left\lfloor \frac{n_i}{2} \right\rfloor$$

where m is the number of graphs and n_i is the number of vertices on the i^{th} graph.

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Removal of One line in the Positive-Triangle-Wins game

Let a game be played on n >3 vertices, as illustrated. Players A and B take turns labeling an edge on K_n, in this case K_7. The winner of the game is whoever manages to label an edge such that a positive triangle is formed.

Let any edge be removed from the graph (1). This is illustrated by an oval connecting two vertices, as opposed to a linear edge. There are now two cases for A_1.

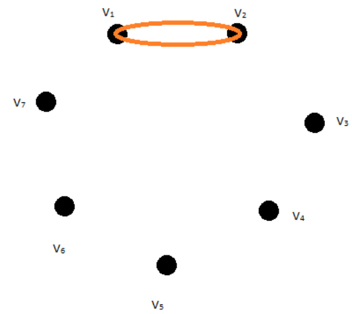


Figure 1

Case 1: A_1 is coincident to the missing edge (2a). This creates two cases for B_1:

Case a: B_1 is played to form a triangle between A_1, B_1, and the missing edge (2b). In this case, A_2 cannot be coincident to v_2, v_1 or v_7, else a positive triangle could easily be formed the next turn by B. Therefore, A_2 must only be coincident to v_3, v_4, v_5, or v_6. No further moves in the game may be coincident to v_1, v_2, or v_7 either. Therefore, if A_1 is coincident to the missing edge, and B_1 forms a triangle with A_1, B_1 and the missing edge, this essentially creates a new game of K_{n-3}, in this case K_4 (indicated by the circle). Player A gets to start this new game, so B would win.

Case b: B_1 is not played to form a triangle between A_1, B_1, and the missing edge (2c). Therefore, a new game of K_5 has just been started on v_2, v_3, v_4, v_5, and v_6 (as indicated by the circle). No edges may be played on v_1 or v_7 because that would clearly lead to a loss the very next turn. In addition, we have already covered the consequences of and edge between v_2 and v_7 in Case 1. In all K_5 games, the first player to play under optimal strategy will win. Therefore, if A_1 is coincident to the missing edge and B_1 does not create a triangle with A_1 and the missing edge, B will win.

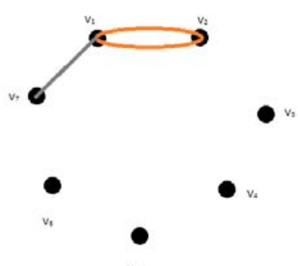


Figure 2a

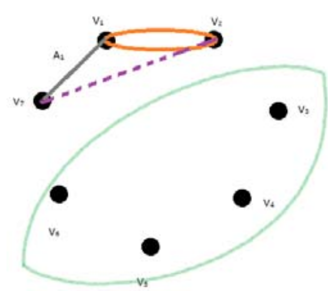


Figure 2b

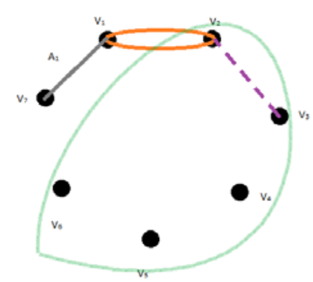


Figure 2c

Case 2: A_1 is not coincident to the missing edge (3). As soon as A_1 is played, this effectively creates a game R_{n-2} (as indicated by the circle). It is impossible to immediately ascertain the winner of this game, but we know that Player B for the original R_n game is now Player A for this new R_{n-2}. Thus, we have a recursive formula for the winner of R_n when the first move is not coincident to the missing edge. In the given example, labeling an edge non-coincident to the missing edge turns the R_7 game into an R_5 game (as indicated by the circle).

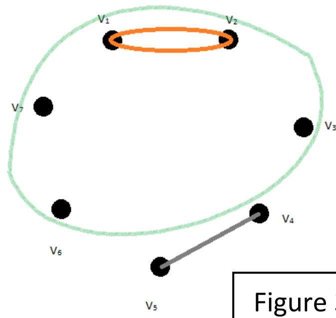


Figure 3

Cases n = 3, 4 are quite trivial. When n = 3, removal of one edge creates a graph with two edges. No triangle can be formed. In n = 4, A's strategy is to always play opposite to the missing edge. B's only remaining moves involve playing an edge coincident to A_1, so A can easily win the very next turn.

What this means is that knowledge of K_n for all values (already discovered and proven above), and knowledge of R_n for values 3, 4, and 5 (very easy to find) will give the winner of every game of R_n. Player A can only win if either Case 1 or Case 2 is a guaranteed win for him. Player B will only win if both Case 1 and Case 2 indicate B victory.

Part B

Player A Ensured Victory in K_6 using Complexes:

Notes

1. A complex is formed when a quadrilateral's sides are labeled CCCD, with C and D being (-) or (+), the two labels available for this game. The labels possible for any three coincident labeled edges (P_4) are CCC, CDC, CCD, DCC. In all of these cases, a single edge between the two farthest vertices will be able to create a complex, by labeling the edge either A or B, depending on the situation. Thus, any labeled P_4 sets up for an immediate complex, assuming no outside restrictions. The fewest outside restrictions occur in the beginning of a game, when few edges are labeled.
2. Conjecture: If any complex is formed in the game, every possible complex will be formed in that game. In K_6, this creates three complexes which prevent labeling on 4 edges. Since there are 15 total edges on K_6, this means that only 11 edges will be available for labeling. Thus, if any complex is formed, all complexes will be formed, there will be an odd number of available edges, and Player A will win because they will have the last available edge to label.

Player A starts play, drawing any edge (1). Note: Diagram colors do not represent positive or negative labeling; they only represent players. Red indicates player A, blue indicates player B, and a dotted dashed line indicates a potential move, and will be explained.

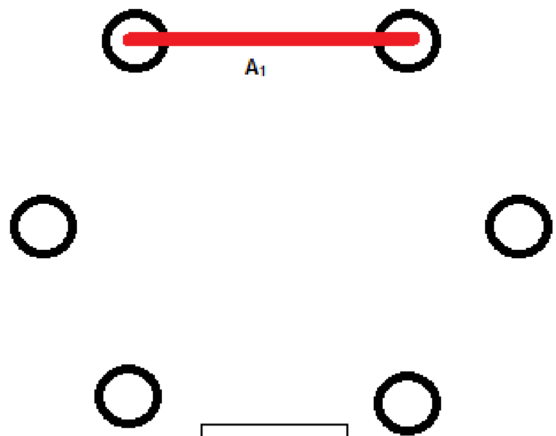


Figure 1

Player B now has two cases for their play:

Case 1: B_1 is coincident to A_1 (2a)

In this case, optimal strategy for A_2 would be to label an edge that is not coincident to either of the existing labeled edges (2b). For B_2, they again have two cases which do not create some P_4.

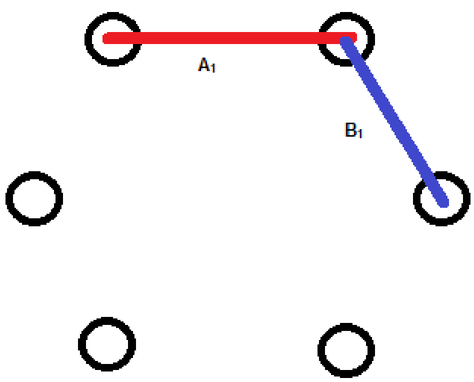


Figure 2a

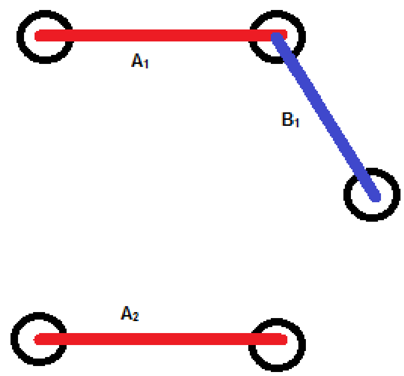


Figure 2b

Case a: B_2 is coincident to either A_1 or B_1 (2c). In this case, this creates some P_4, composed of A_1, B_1, and B_2. Thus, a complex may be formed by Player A on the next turn. Therefore, all possible complexes will be formed, and Player A will win.

Case b: B_2 is not coincident to A_1 or B_1 (2d). This means that B_2 must be coincident to A_2. In this case, A_1 and B_1 would be coincident, and A_2 and B_2 would be coincident. In this case, optimal strategy for A_3 would be to label the edge between v_2 and v_5 (2e). This creates multiple P_4s, and no matter where B places their next move, some complex will be formed, and thus A will win.

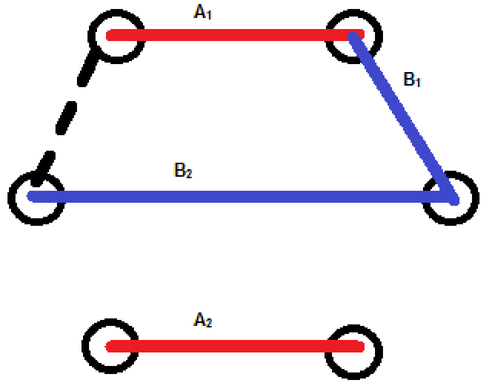


Figure 2c

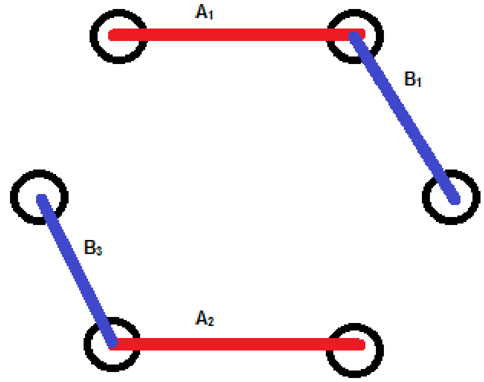


Figure 2d

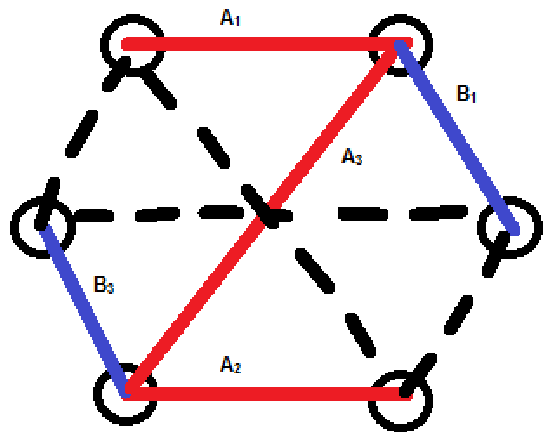


Figure 2e

Case 2: B_1 is not coincident to A_1 (3a)

In this case, optimal strategy for A_2 would be to label an edge non-coincident to both A_1 and B_1 (3b). Since every edge in a graph connects two vertices, and this game on 6 vertices already has 2 non-coincident edges, drawing a third non-coincident means that on the next turn, B_2 will create some P_4. Thus, A will be able to create a complex, and therefore win.

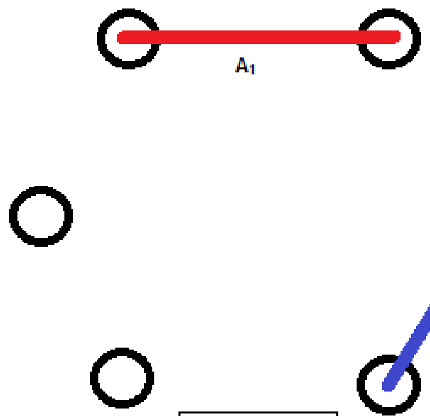


Figure 3a

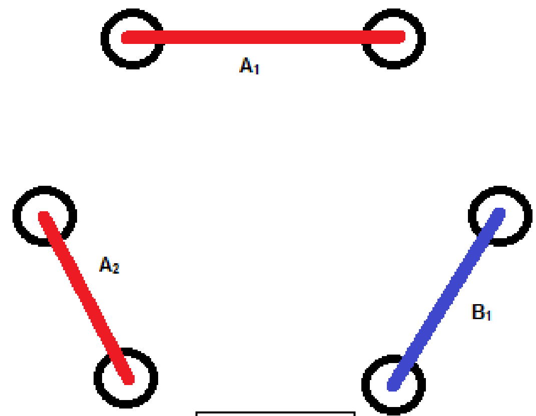


Figure 3b

In conclusion, Player A will always win a Positive-Triangle-Loses Game on K_6. Every game will start with one of the above 4 sub cases, and in every case A will be able to form a complex, thus guaranteeing the win.

Note player A will be able to force a complex in the same manner for all games where n >5. The exact same cases still apply, but only the numbering of the vertices will change. This means that player A can guarantee the formation of a complex in any game. However, this may or not be beneficial to him.

- -

Proof that K5 graph will always end in a draw

Optimal Strategy for Player A: Create a complex.

The maximum number of complexes in a K_5 is 2. If a complex is created, then a double complex will be created inevitably. This takes away 3 of 10 moves, leaving 7 safe moves without making a positive triangle. This means the first player will take the last safe move.

Optimal Strategy for Player B: Create a draw.

If a complex is created, A wins, so B's optimal strategy is to prevent a complex from being formed. Unless A does not play according to his/her own optimal strategy, B will not win, because the formation of any complex will cause the formation of all possible complexes, therefore eliminating three of the ten available moves, letting A have the last free move.. Thus, the optimal strategy for B is to create a draw.

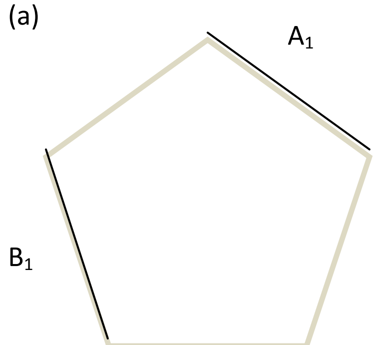
The K_5 Graph

In K_5, it is possible for B to prevent all complexes from being formed. When A makes the first move, B must not make a move coincident to A_1 (a).

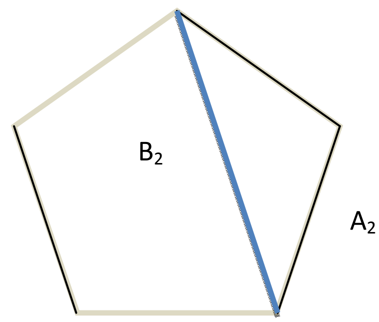
Case 1: A_2 move coincident to one line.

B must create a triangle out of the two connected lines (1a). This forces A to make the third edge of a complex. Since a complex is created by four lines, B needs to draw the last line in such a way that a complex is not formed.

(a)

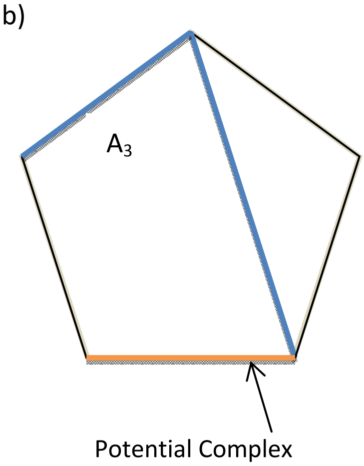


(1a)

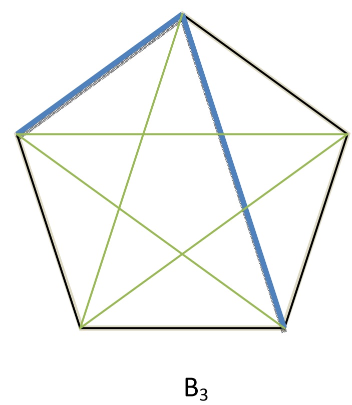


When A creates the third line for the potential complex (1b), B should then ensure that there is an even number of negative signs (0, 2, or 4) on the quadrilateral to diffuse the complex (1c). The other potential complex will have been diffused in the process, resulting in a forced draw (1d).

(1b)

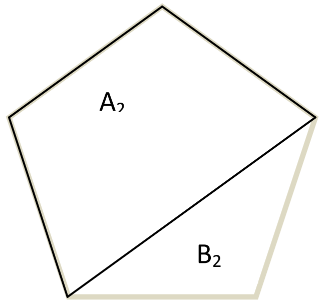


(1c)

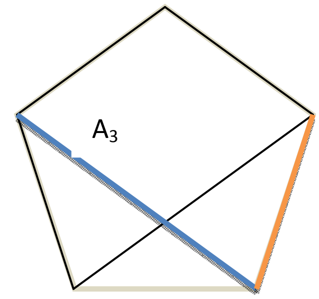


<u>Case 2:</u> A_2 connects the two formed lines.
This creates a P_4. B must then prevent the complex by making a move such that an even number of negative signs compose the quadrilateral (2a).
Now, A can only attempt to create a new potential complex (2b), which can once again be diffused by B in the same way (2c). This will once again result in a draw.

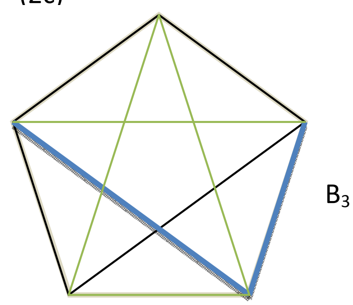
(2a)



(2b)



(2c)



<u>Conclusion</u>

There are few enough nodes and possible moves that, although A originally has the advantage, B can alleviate all of the attempts to create a complex.
Since only two complexes can be created on the K_5 graph, B is able to approach them and diffuse them, an option not available in K_6 or higher. In those cases, too many nodes, potential moves, and potential complexes are in place for the predicted loser to diffuse all of them.

- -

Winner of P-T-Loses game on K_4

The winner of a P-T-Loses game on K_4 is actually quite trivial. No matter where A plays, B should simply label the edge non-coincident to A_1 (1a). There will only be one such edge. Thus, A_2 must be coincident to both A_1 and B_1 (1b). Therefore, B_2 can create a complex (1c). Since 4 moves were used to create the complex, and 2 moves were eliminated, B has taken the last free move, and will therefore win K_4. Therefore, B will <u>always</u> win a P-T-Loses game on K_4.

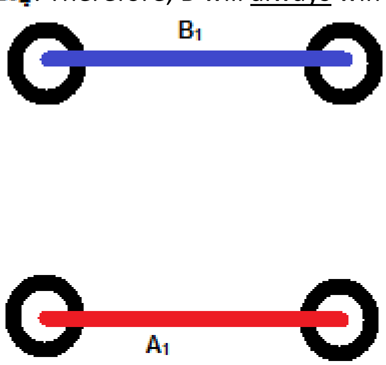


Figure 1a

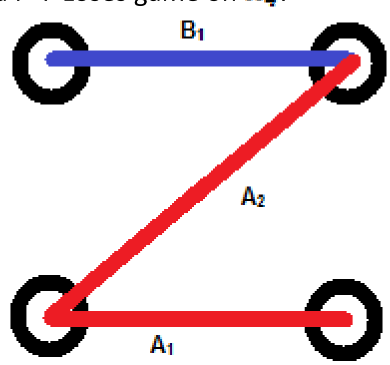


Figure 1b

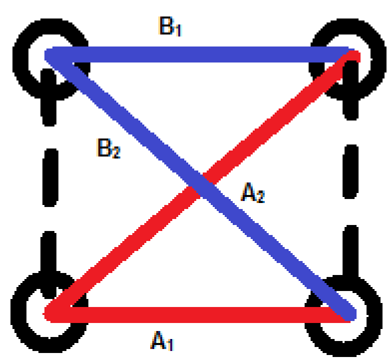


Figure 1c

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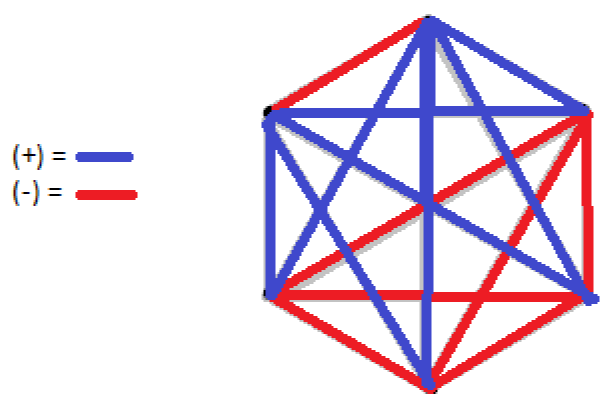
Part C

The Existence of Draws & the Mathematical Theory of Additive Graphs

Note- violates optimal strategy for Part A (and in certain cases for Part B)
The only types of triangles that can exist in a draw game are those with an odd number of negatives: (+,+,-) or (-,-,-)

- Let n_1 and n_2 exist such that $n_1 + n_2 = n$, where n is the number of vertices in a completed graph
- To find draw games, denote values to n_1 and n_2 for the desired n
- Create complete graphs for K_{n1} and K_{n2} independently, using only negative edges
- Connect all vertices of K_{n1} and K_{n2}

Ex: n =6, n_1=2, and n_2 =4



Note how one set of red lines form a K_2 and the other a K_4.

Consider the following cases for triangle in K_n:

- Case 1: a triangle exists entirely inside of K_{nx}. The triangle is thus composed of entirely negative edges, and there is no positive triangle.
- Case 2: the triangle stretches across K_{n1}, and K_{n2}. Since the 'base' of the triangle only exists in one K_{nx}, and the connecting edges between K_{n1} and K_{n2}, are positive, these triangles will always be (+,+,-), and thus not positive

Therefore, if some K_n is divided into K_{n1} and K_{n2}, it will be a draw game.

However, since a draw game must only be composed of (-,-,-) and (+,+,-) triangles, this necessitates that every draw game can have its vertices rearranged in order to fit this pattern.

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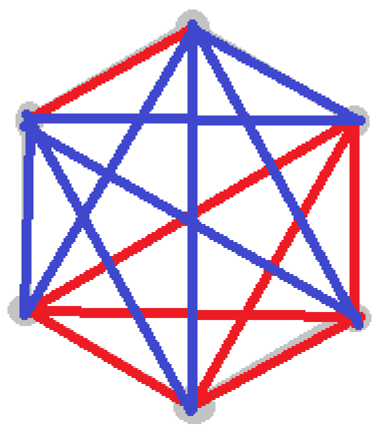
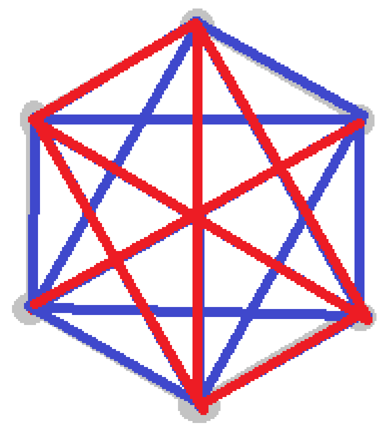


is analogous to



Every draw game must take on this form of two completed negative graphs with vertices connected by positive segments.

Seen in the diagram above:

- if any positive line had its label switched , a (+,-,-) triangle forms
- if any negative line switches label, a (+,+,+) triangle forms

Therefore, the number of draw games for any graph on n vertices is the number of ways to have a pair of additives add up to n. The number of draws is therefore $\left\lfloor \frac{n}{2} \right\rfloor + 1$.